Math 4650 - Homework # 2
Sequences

Part 1 - Computations

1. For $\epsilon = 0.001$, find an integer N such that if $n \geq N$ then $\left|\frac{1}{\sqrt{n}} - 0\right| < \epsilon$. Draw a picture.

2. For $\epsilon = 0.01$, find and integer N such if that $n \geq N$ then $\left|\frac{2n}{3n+1} - \frac{2}{3}\right| < \epsilon$. Draw a picture.

Part 2 - Proofs

3. (a) Use the ϵ-definition of limit to show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

   (b) Use the ϵ-definition of limit to show that $\lim_{n\to\infty} \frac{n+2}{5n-3} = \frac{1}{5}$.

   (c) Use the ϵ-definition of limit to show that $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$

   (d) Use the ϵ-definition of limit to show that $\lim_{n\to\infty} n^4$ does not exist.

   (e) Use the ϵ-definition of limit to show that $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$.

   (f) Use the ϵ-definition of limit to show that if $0 < r < 1$ then $\lim_{n\to\infty} r^n = 0$.

   (g) Use the ϵ-definition of limit to show that $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n!} = 0$.

4. Let (a_n) and (b_n) be convergent sequences that converge to A and B, respectively. Let $\alpha \neq 0$ and $\beta \neq 0$ be a real numbers.

   Prove the following using the definition of limit of a sequence.

   (a) Prove that the sequence (αa_n) converges to αA.

   (b) Prove that the sequence $(\alpha a_n + \beta b_n)$ converges to $\alpha A + \beta B$.

   (c) Prove that the sequence $(\alpha_n \beta_n)$ converges to AB.

5. (Squeeze Theorem) Suppose that (a_n), (b_n), and (c_n) are sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all n. If both (a_n) and (c_n) both converge to L, then (b_n) converges to L.

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6. Prove (a) and then use (a) to prove (b) and (c).

   (a) Prove that if (x_n) is a convergent sequence of real numbers where $x_n \geq 0$ for all n and $\lim_{n\to\infty} x_n = L$, then $L \geq 0$.

   (b) Suppose that (a_n) and (b_n) are convergent sequences of real numbers such that $a_n \leq b_n$ for all n. Prove that if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then $A \leq B$.

   (c) Suppose that (a_n) is a convergent sequence of real numbers. Prove that if $C \leq a_n \leq D$ for all n, then $C \leq \lim_{n\to\infty} a_n \leq D$.

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7. Let (a_n) be a convergent sequence of real numbers. Suppose that $\lim_{n\to\infty} a_n = L$ where $L \neq 0$. Prove that there exists $M > 0$ and $N > 0$ where if $n \geq N$ then $|a_n| \geq M$.

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8. Let (a_n) be a convergent sequence with $a_n \to L$. Prove that any subsequence (a_{n_k}) must also converge to L.

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9. Suppose that (a_n) is a Cauchy sequence. Using the definition of Cauchy sequence, prove that (a_n) is a bounded sequence.
